Laplace Transform

The French Newton
Pierre-Simon Laplace

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
- One of the first scientists to suggest the existence of black holes

History of the Transform

- Euler began looking at integrals as solutions to differential equations in the mid 1700's:

$$z = \int X(x)e^{ax}\,dx$$

$$z(x;a) = \int_0^x e^{at}\,X(t)\,dt,$$

- Lagrange took this a step further while working on probability density functions and looked at forms of the following equation:

$$\int X(x)e^{-ax}a^x\,dx,$$

- Finally, in 1785, Laplace began using a transformation to solve equations of finite differences which eventually lead to the current transform

$$S = Ay_s + B\,\Delta y_s + C\,\Delta^2 y_s + \ldots,$$

$$y_s = \int e^{-sx}\,\phi(x)\,dx,$$

Definition

- The Laplace transform is a linear operator that switched a function f(t) to F(s).
- Specifically: $F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$

  where: $s = \sigma + i\omega.$
- Go from time argument with real input to a complex angular frequency input which is complex.

Restrictions

- There are two governing factors that determine whether Laplace transforms can be used:
  - f(t) must be at least piecewise continuous for $t \geq 0$
  - $|f(t)| \leq Me^{\gamma t}$ where M and γ are constants

Continuity

- Since the general form of the Laplace transform is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

  it makes sense that f(t) must be at least piecewise continuous for t ≥ 0.
- If f(t) were very nasty, the integral would not be computable.

Boundedness

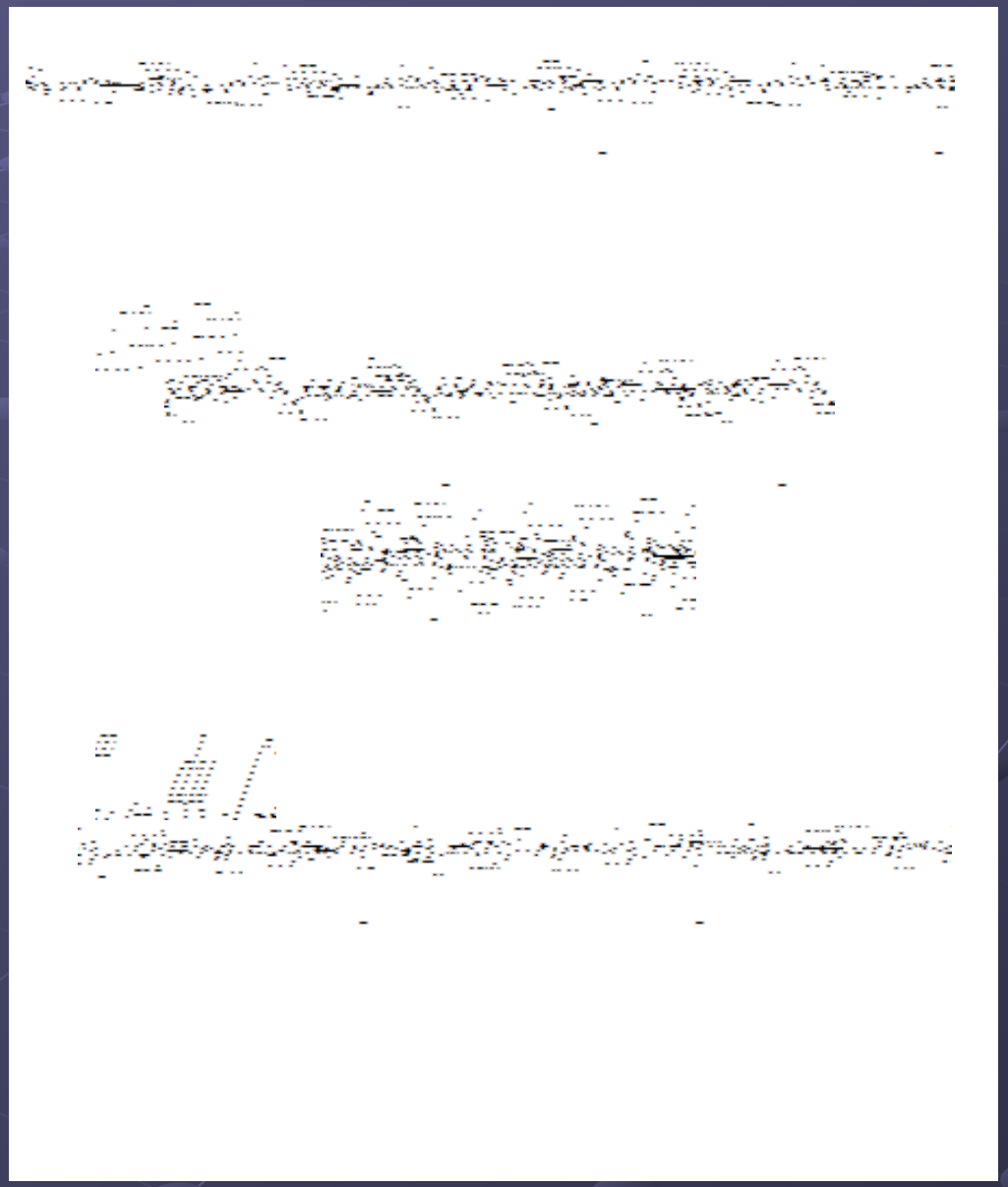
- This criterion also follows directly from the general definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

- If f(t) is not bounded by $Me^{\gamma t}$ then the integral will not converge.

Laplace Transform Theory

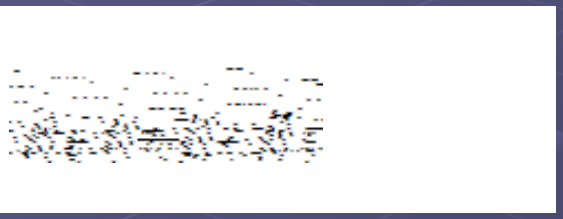
- General Theory
- Example
- Convergence



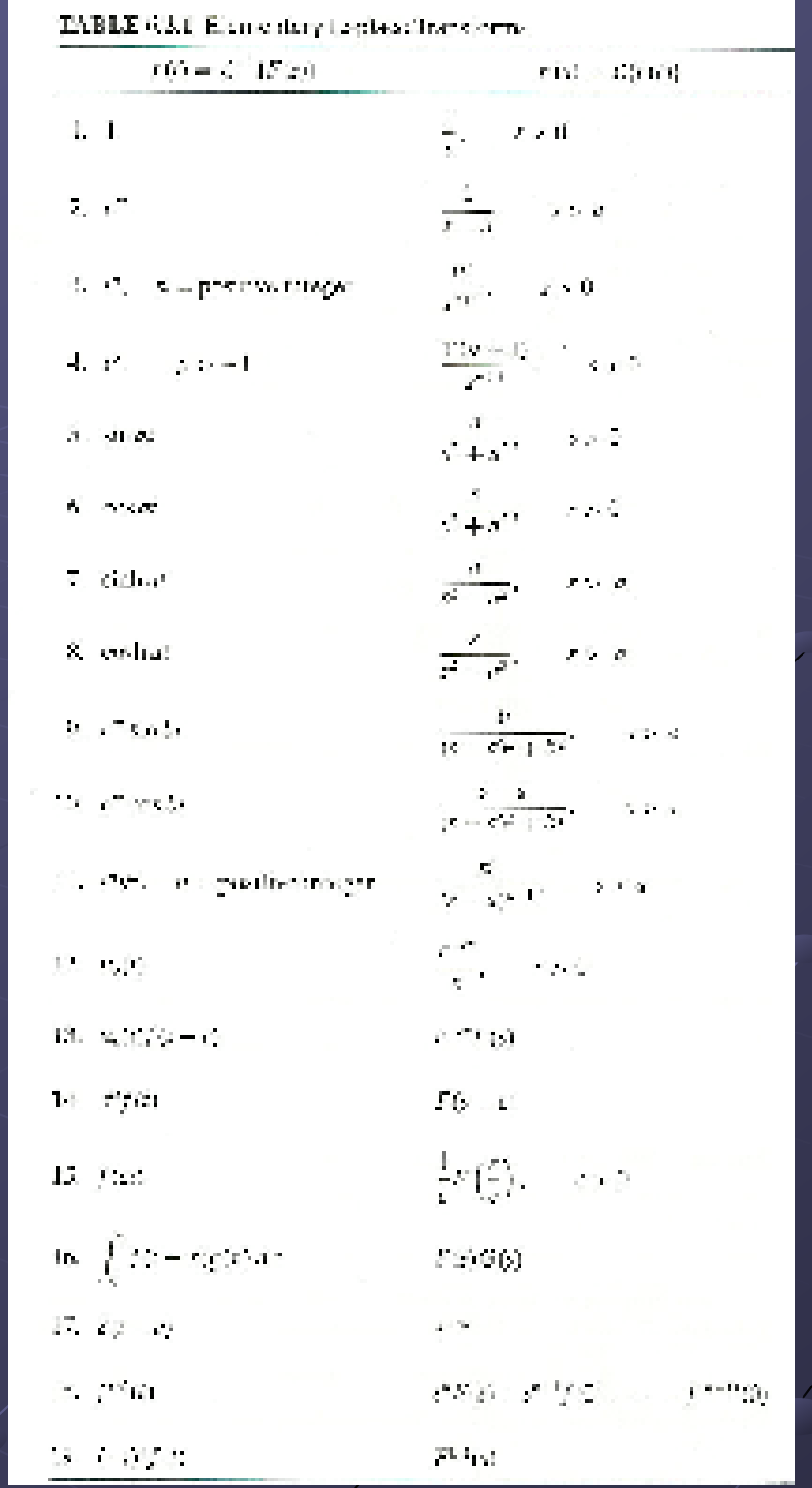
Laplace Transforms

- Some Laplace Transforms
- Wide variety of function can be transformed

- Inverse Transform

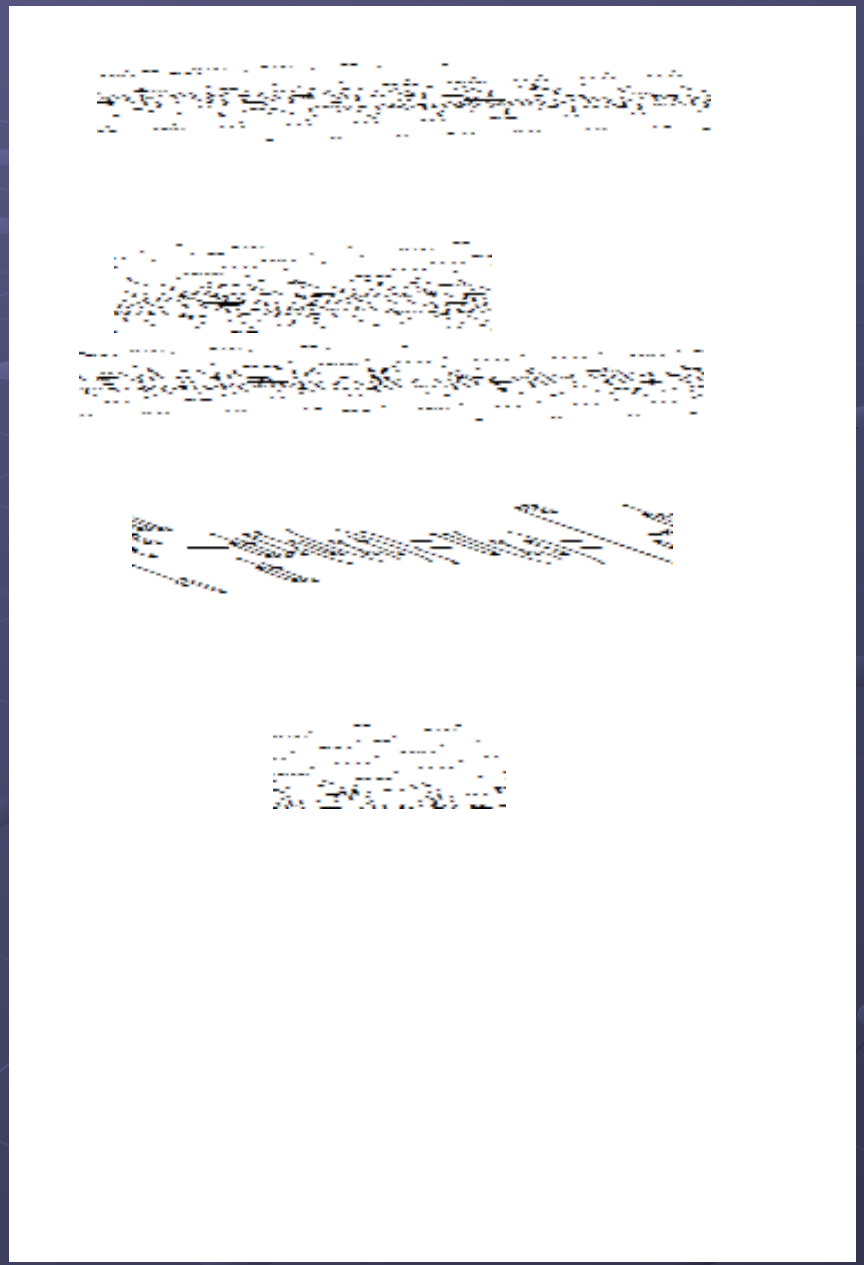


- Often requires partial fractions or other manipulation to find a form that is easy to apply the inverse



Laplace Transform for ODEs

- Equation with initial conditions
- Laplace transform is linear
- Apply derivative formula
- Rearrange
- Take the inverse

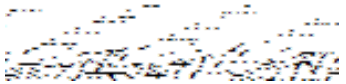


Laplace Transform in PDEs

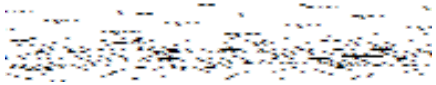
Laplace transform in two variables (always taken with respect to time variable, t):

$$\mathcal{L}\{u(x,t)\} = U(x,s) = \int_0^\infty e^{-st}\frac{du}{dt}\,dt$$

Inverse laplace of a 2 dimensional PDE:



Can be used for any dimension PDE:



The Transform reduces dimension by “1”:

- ODEs reduce to algebraic equations
- PDEs reduce to either an ODE (if original equation dimension 2) or another PDE (if original equation dimension >2)

Consider the case where:

$u_x+u_t=t$ with $u(x,0)=0$ and $u(0,t)=t^2$ and

Taking the Laplace of the initial equation leaves $U_x+ U=1/s^2$ (note that the partials with respect to "x" do not disappear) with boundary condition $U(0, s)=2/s^3$

Solving this as an ODE of variable x, $U(x,s)=c(s)e^{-x} + 1/s^2$

Plugging in B.C., $2/s^3=c(s) + 1/s^2$ so $c(s)=2/s^3 - 1/s^2$

$U(x,s)=(2/s^3 - 1/s^2) e^{-x} + 1/s^2$

Now, we can use the inverse Laplace Transform with respect to s to find

$u(x,t)=t^2e^{-x} - te^{-x} + t$

Example Solutions

Diffusion Equation

$u_t = ku_{xx}$ in (0,l)

Initial Conditions:

$u(0,t) = u(l,t) = 1$, $u(x,0) = 1 + \sin(\pi x/l)$

Using $af(t) + bg(t) \rightarrow aF(s) + bG(s)$

and $df/dt \rightarrow sF(s) - f(0)$

and noting that the partials with respect to x commute with the transforms with respect to t, the Laplace transform U(x,s) satisfies

$sU(x,s) - u(x,0) = kU_{xx}(x,s)$

With $e^{at} \rightarrow 1/(s-a)$ and a=0,

the boundary conditions become U(0,s) = U(l,s) = 1/s.

So we have an ODE in the variable x together with some boundary conditions. The solution is then:

$U(x,s) = 1/s + (1/(s+k\pi^2/l^2))\sin(\pi x/l)$

Therefore, when we invert the transform, using the Laplace table:

$u(x,t) = 1 + e^{-k\pi 2t/l2}\sin(\pi x/l)$

Wave Equation

$u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$

Initial Conditions:

$u(0,t) = f(t)$, $u(x,0) = u_t(x,0) = 0$

For $x \to \infty$, we assume that $u(x,t) \to 0$. Because the initial conditions vanish, the Laplace transform satisfies

$s^2 U = c^2 U_{xx}$

$U(0,s) = F(s)$

Solving this ODE, we get

$U(x,s) = a(s)e^{-sx/c} + b(s)e^{sx/c}$

Where a(s) and b(s) are to be determined.

From the assumed property of u, we expect that $U(x,s) \to 0$ as $x \to \infty$.

Therefore, $b(s) = 0$. Hence, $U(x,s) = F(s)\, e^{-sx/c}$. Now we use

$H(t-b)f(t-b) \to e^{-bs}F(s)$

To get

$u(x,t) = H(t - x/c)f(t - x/c)$.

Notation

- R = ratio of induced electric field to the product of the current density and the applied magnetic field
- ρ = electrical resistance
- H = magnetic field
- J = current density
- E = applied electric field
- n = concentration of electrons
- u_l = mobility

$$E_x = \rho J_x - RHJ_y$$

$$E_y = RHJ_x + \rho J_y$$